A Loxodrome in the Web-Mercator Projection

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Map projections are mathematical procedures that enable the mapping of the earth's or other celestial bodies' curved surface to a plane.

The theory of map projections is often referred to as the mathematic cartography.

The goal of studying map projections is the creation of mathematical basis for making maps and solving theoretic and practical problems in cartography, geodesy, geography, astronomy, navigation and other related sciences.

http://ica-proj.kartografija.hr/home.en.html

1. Parameterizations of a sphere and an ellipsoid

2. Map projections and their linear scales
Mercator projection of a sphere
Mercator projection of an ellipsoid
Web-Mercator projection

3. Loxodrome and its image in the plane of projection
Mercator projection of a sphere
Mercator projection of an ellipsoid
Web-Mercator projection

4. Comparison of the images of loxodrome in the Mercator and Web-Mercator projections

123 equations

Geographic parameterization of a sphere

$$x = R\cos\varphi\cos\lambda$$

$$y = R\cos\varphi\sin\lambda$$

$$z = R\sin\varphi$$

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \lambda \in [-\pi, \pi]$$

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2$$

Geodetic parameterization of a rotational ellipsoid

$$x = N\cos\varphi\cos\lambda$$

$$y = N\cos\varphi\sin\lambda$$

$$z = N(1-e^2)\sin\varphi \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \lambda \in [-\pi, \pi]$$

$$e^2 = \frac{a^2 - b^2}{a^2} \qquad N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \qquad M = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}}$$

$$ds^2 = M^2 d\varphi^2 + N^2\cos^2\varphi d\lambda^2$$

Mapping a sphere or an ellipsoid into a plane = Map projection

$$x = x(\varphi, \lambda) \qquad y = y(\varphi, \lambda)$$

$$E = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 \qquad F = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} \qquad G = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2$$

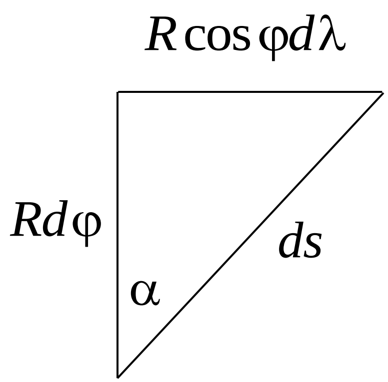
$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2$$

Linear scale c for mapping a sphere

$$c^2 = \frac{ds'^2}{ds^2} = \frac{Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2}{R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2} = \frac{E}{R^2}\cos^2\alpha + \frac{F}{R^2\cos\varphi}\sin 2\alpha + \frac{G}{R^2\cos\varphi}\sin^2\alpha$$

where

$$\tan\alpha = \frac{\cos\varphi d\lambda}{d\varphi}$$

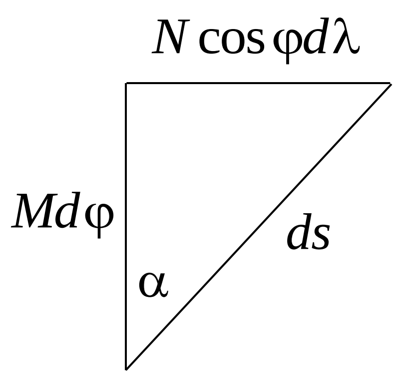


Linear scale c for mapping an ellipsoid

$$c^2 = \frac{ds'^2}{ds^2} = \frac{Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2}{M^2 d\varphi^2 + N^2\cos^2\varphi d\lambda^2} = \frac{E}{M^2}\cos^2\alpha + \frac{F}{MN\cos\varphi}\sin 2\alpha + \frac{G}{N^2\cos^2\varphi}\sin^2\alpha$$

where

$$\tan\alpha = \frac{N\cos\varphi d\lambda}{Md\varphi}$$



Mercator projection of a sphere, normal aspect

$$x = K\lambda \qquad y = K \operatorname{Ar}\tanh(\sin\varphi) \qquad K = const. > 0$$

$$\frac{\partial x}{\partial \varphi} = 0 \qquad \frac{\partial x}{\partial \lambda} = K \qquad \frac{\partial y}{\partial \varphi} = \frac{K}{\cos\varphi} \qquad \frac{\partial y}{\partial \lambda} = 0$$

$$E = \frac{K^2}{\cos^2\varphi} \qquad F = 0 \qquad G = K^2$$

$$c^2 = \frac{K^2}{R^2\cos^2\varphi}\cos^2\alpha + \frac{K^2}{R^2\cos^2\varphi}\sin^2\alpha = \frac{K^2}{R^2\cos^2\varphi}$$

$$c = c(\varphi) = \frac{K}{R\cos\varphi} \qquad \rightarrow \text{Conformality}$$

Mercator projection of an ellipsoid, normal aspect

$$x = K\lambda \qquad y = K\left[\operatorname{Ar}\tanh(\sin\varphi) - e\operatorname{Ar}\tanh(e\sin\varphi)\right] \qquad K = const. > 0$$

$$\frac{\partial x}{\partial \varphi} = 0 \qquad \frac{\partial x}{\partial \lambda} = K \qquad \frac{\partial y}{\partial \varphi} = \frac{K(1-e^2)}{\cos\varphi(1-e^2\sin^2\varphi)} \qquad \frac{\partial y}{\partial \lambda} = 0$$

$$E = \frac{K^2(1-e^2)^2}{\cos^2\varphi(1-e^2\sin^2\varphi)^2} = \frac{K^2M^2}{N^2\cos^2\varphi} \qquad F = 0 \qquad G = K^2$$

$$c^2 = \frac{K^2}{N^2\cos^2\varphi}\cos^2\alpha + \frac{K^2}{N^2\cos^2\varphi}\sin^2\alpha = \frac{K^2}{N^2\cos^2\varphi}$$

$$c = c(\varphi) = \frac{K}{N\cos\varphi} \qquad \rightarrow \text{Conformality}$$

Web-Mercator projection of an ellipsoid

$$x = a\lambda \qquad y = a\operatorname{Ar}\tanh(\sin\varphi)$$

$$\frac{\partial x}{\partial \varphi} = 0 \qquad \frac{\partial x}{\partial \lambda} = a \qquad \frac{\partial y}{\partial \varphi} = \frac{a}{\cos\varphi} \qquad \frac{\partial y}{\partial \lambda} = 0$$

$$E = \frac{a^2}{\cos^2\varphi} \qquad F = 0 \qquad G = a^2$$

$$c^2 = \frac{a^2}{M^2\cos^2\varphi}\cos^2\alpha + \frac{a^2}{N^2\cos^2\varphi}\sin^2\alpha = \frac{a^2}{\cos^2\varphi}\left(\frac{\cos^2\alpha}{M^2} + \frac{\sin^2\alpha}{N^2}\right)$$

$$c = c(\varphi, \alpha) \qquad \rightarrow \text{Non-conformality}$$

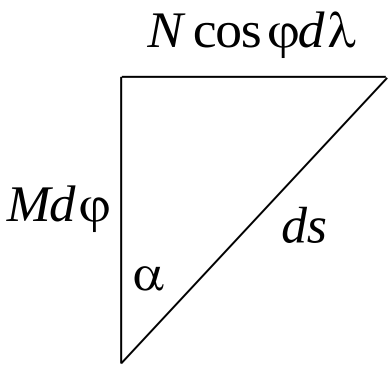
$$\alpha = 0 \qquad c = \frac{a}{M\cos\varphi}$$

$$\alpha = \frac{\pi}{2} \qquad c = \frac{a}{N\cos\varphi}$$

For a curve on the rotational ellipsoid

$$\cos\alpha\, ds = M d\varphi \qquad \sin\alpha\, ds = N\cos\varphi\, d\lambda \qquad \tan\alpha = \frac{N\cos\varphi\, d\lambda}{M d\varphi}$$

$$ds^2 = M^2 d\varphi^2 + N^2\cos^2\varphi\, d\lambda^2$$



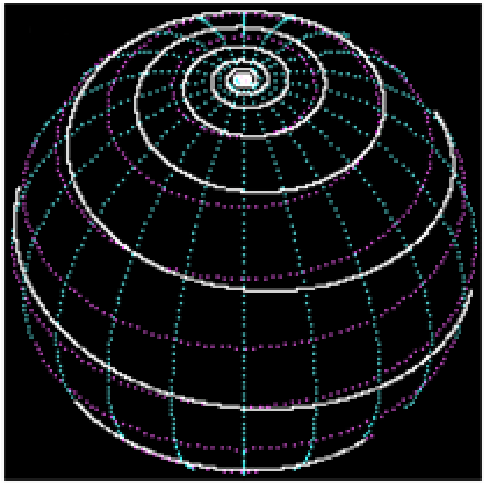
Definition of a loxodrome $\alpha = const.$

$$d\lambda = \tan\alpha \frac{M d\varphi}{N\cos\varphi}$$

$$\lambda = \tan\alpha \int \frac{M d\varphi}{N\cos\varphi} + C = \tan\alpha\; q(\varphi, e) + C$$

where C is a constant of integration

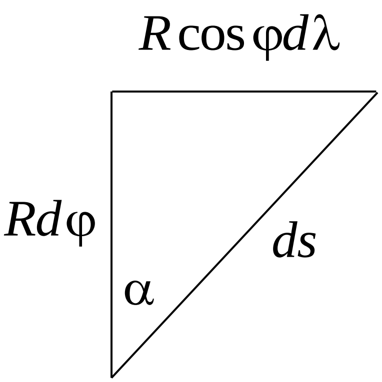
$$q(\varphi, e) = \mathrm{Ar}\tanh(\sin\varphi) - e\,\mathrm{Ar}\tanh(e\sin\varphi)$$



For a curve on the sphere

$$\cos\alpha\, ds = R\, d\varphi \qquad \sin\alpha\, ds = R\cos\varphi\, d\lambda \qquad \tan\alpha = \frac{\cos\varphi\, d\lambda}{d\varphi}$$

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi\, d\lambda^2$$



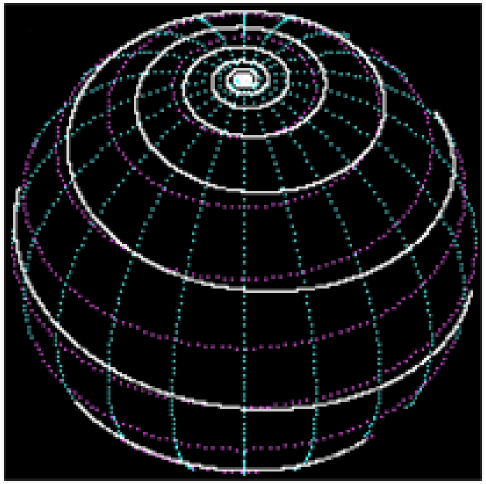
Definition of a loxodrome $\alpha = const.$

$$d\lambda = \tan\alpha \frac{d\varphi}{\cos\varphi}$$

$$\lambda = \tan\alpha \int \frac{d\varphi}{\cos\varphi} + C = \tan\alpha\; q(\varphi) + C$$

where C is a constant of integration

$$q(\varphi) = q(\varphi, 0) = \mathrm{Ar}\tanh(\sin\varphi)$$



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Loxodrome in the Mercator projection of a sphere, normal aspect

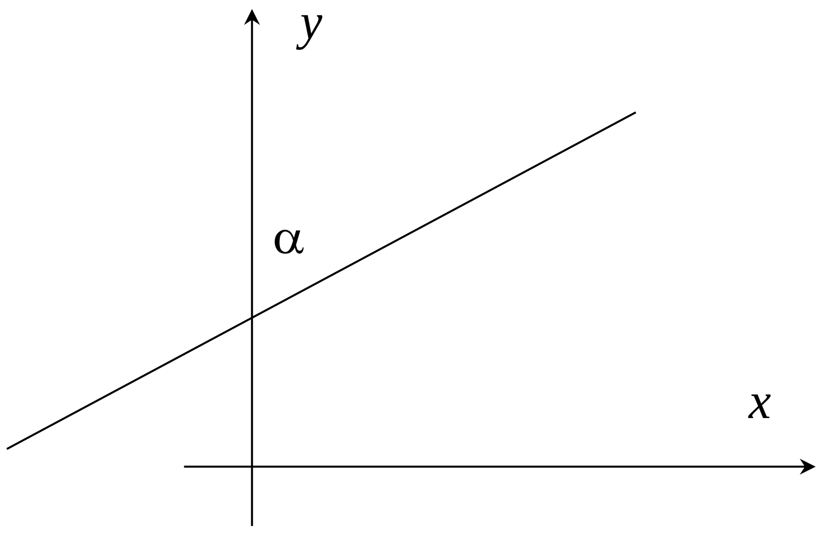
$$x = K\lambda \qquad y = K\operatorname{Ar}\tanh(\sin\varphi) = Kq(\varphi) \qquad K = const. > 0$$

$$\lambda = \tan\alpha\; q(\varphi) + C$$

$$\frac{x}{K} = \tan\alpha\; q(\varphi) + C = \tan\alpha\;\frac{y}{K} + C$$

$$x = \tan\alpha\; y + CK$$

Straight-line

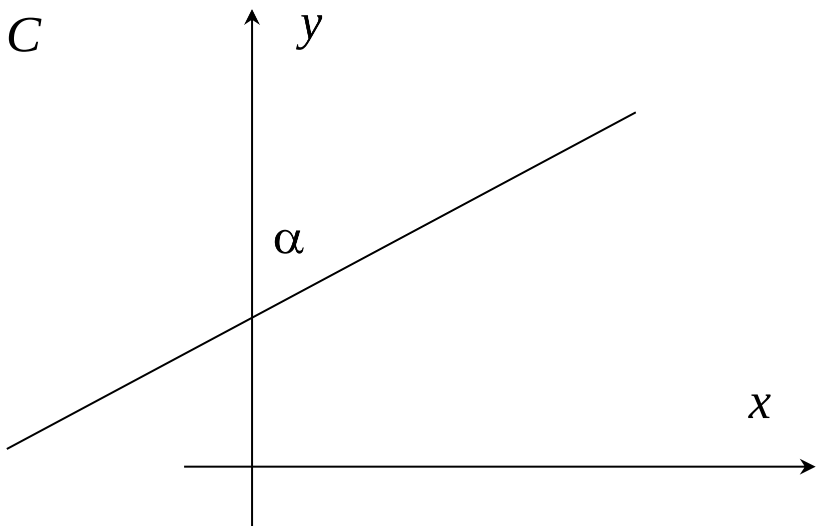


Loxodrome in the Mercator projection of an ellipsoid, normal aspect

$$x = K\lambda \qquad y = K\left[\operatorname{Ar}\tanh(\sin\varphi) - e\operatorname{Ar}\tanh(e\sin\varphi)\right] = Kq(\varphi, e) \qquad K = const. > 0$$

$$\lambda = \tan\alpha\; q(\varphi, e) + C$$

$$\frac{x}{K} = \tan\alpha\; q(\varphi, e) + C = \tan\alpha\,\frac{y}{K} + C$$



$$x = \tan\alpha\; y + CK$$

Straight-line

Image of a loxodrome in the Web-Mercator projection

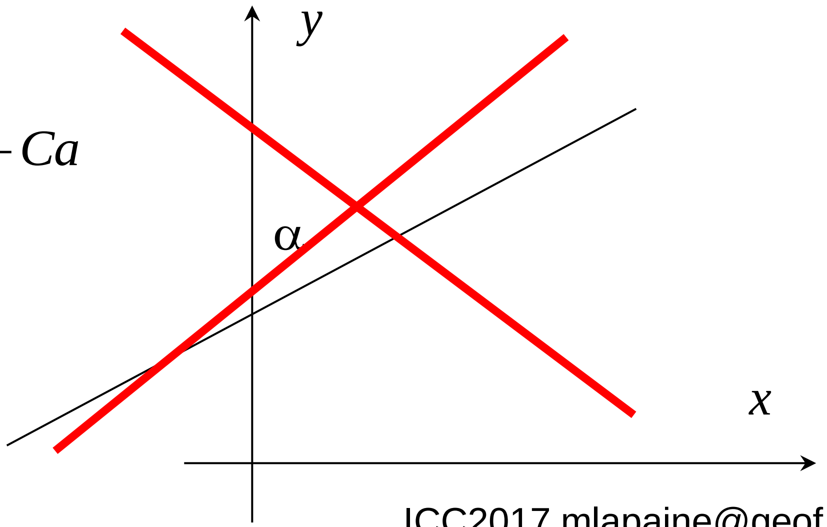
$$x = a\lambda \qquad y = a \operatorname{Ar}\tanh(\sin\varphi) = aq(\varphi) = aq(\varphi, 0)$$

$$\lambda = \tan\alpha \; q(\varphi, e) + C$$

$$\frac{x}{a} = \tan\alpha \; q(\varphi, e) + C = \tan\alpha \left[\operatorname{Ar}\tanh(\sin\varphi) - e\operatorname{Ar}\tanh(e\sin\varphi)\right] + C =$$

$$\tan\alpha\left[\operatorname{Ar}\tanh\left(\tanh\frac{y}{a}\right) - e\operatorname{Ar}\tanh\left(e\tanh\frac{y}{a}\right)\right] + C = \tan\alpha\left[\frac{y}{a} - e\operatorname{Ar}\tanh\left(e\tanh\frac{y}{a}\right)\right] + C$$

$$x = \tan\alpha\left[y - ea\operatorname{Ar}\tanh\left(e\tanh\frac{y}{a}\right)\right] + Ca$$



Not a stright-line, but close to it

Without a loss of generality, we can suppose that our loxodrome goes through the point $(\varphi, \lambda) = (0, 0)$

$$x = \tan\alpha \left[y - ea\operatorname{Ar}\tanh\left(e \tanh\frac{y}{a} \right) \right] = \tan\alpha \left[y - z(y) \right]$$

$$z(y) = ea\operatorname{Ar}\tanh\left(e \tanh\frac{y}{a} \right)$$

$$z(\varphi) = ea\operatorname{Ar}\tanh\left(e\sin\varphi \right) = \frac{ea}{2}\ln\frac{1+e\sin\varphi}{1-e\sin\varphi}$$

$$\frac{dz}{d\varphi} = \frac{ae^2\cos\varphi}{1-e^2\sin^2\varphi} \geq 0$$

$z = z(\varphi)$ is monotono increasing function

The difference between the image of a loxodrome in the Mercator projection of the ellipsoid and the image of the same loxodrome in the Web-Mercator can be illustrated in this way

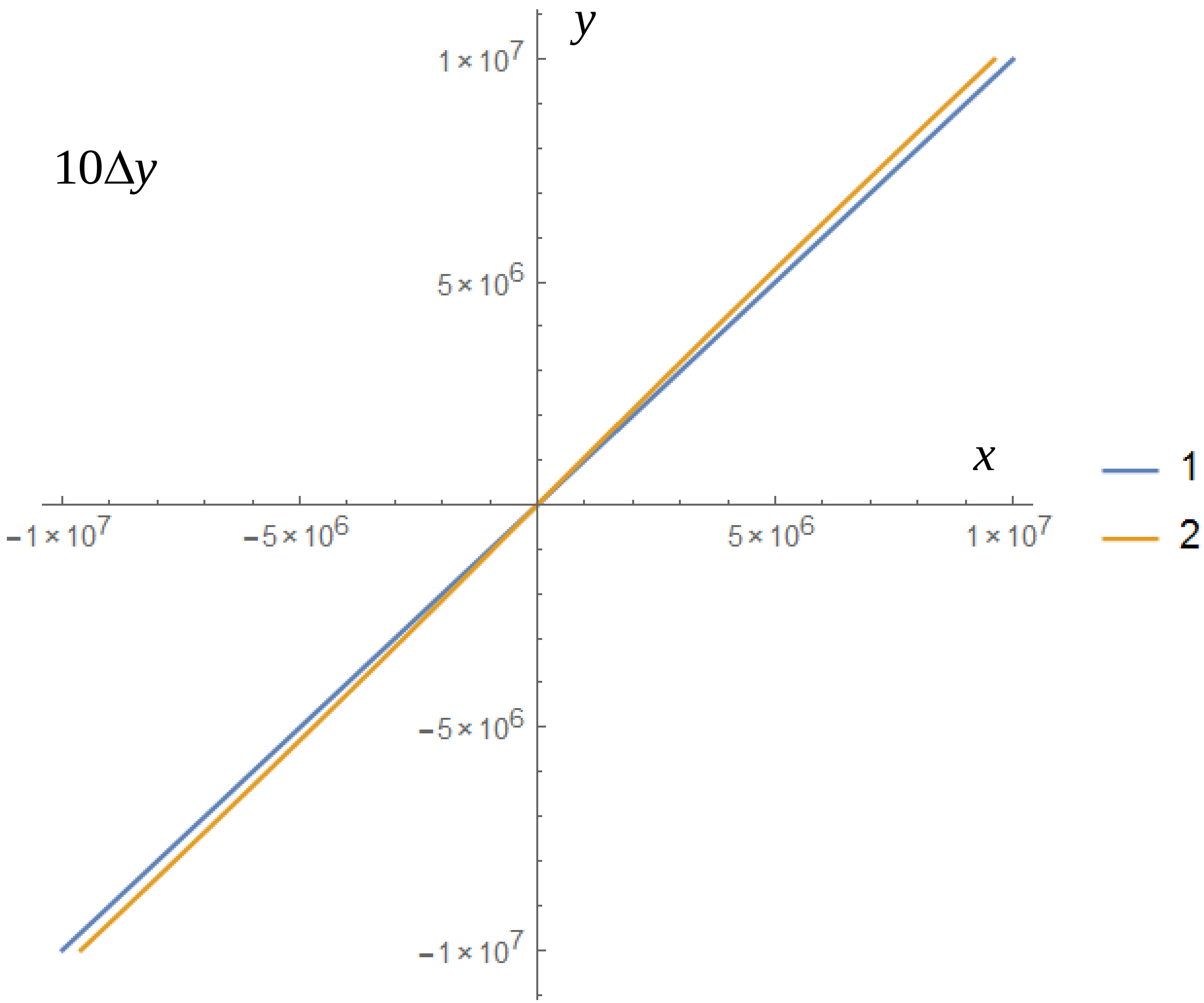
$$\Delta y = \tan\alpha\ y - \tan\alpha\left[y - z(y)\right] = \tan\alpha\ z(y) = \tan\alpha\,\frac{ea}{2}\ln\frac{1+e\sin\varphi}{1-e\sin\varphi}$$

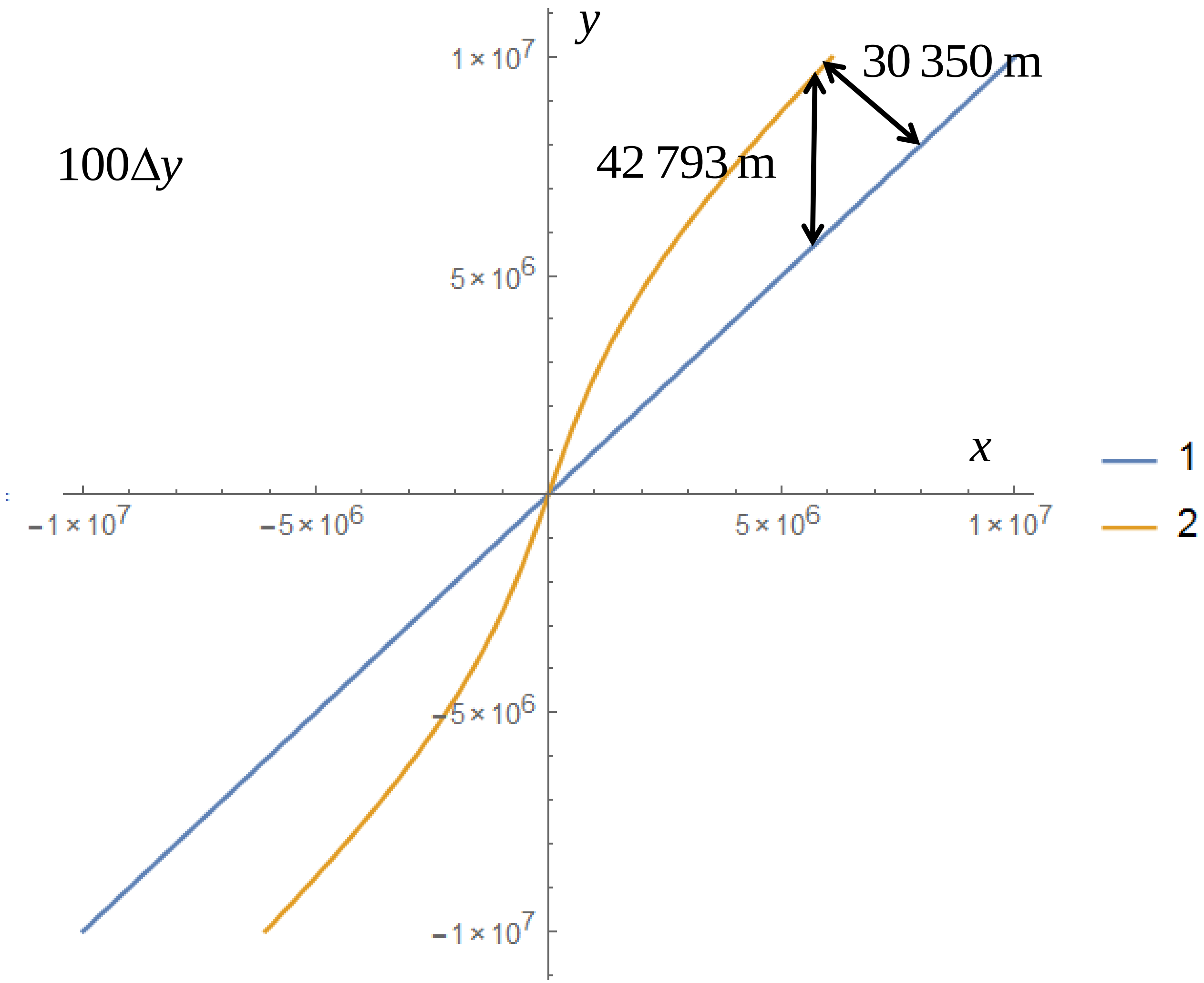
$$\varphi = 0 \quad \Rightarrow \quad \Delta y = 0$$

$$\max\left|\Delta y\right| = \left|\tan\alpha\right|\frac{ea}{2}\ln\frac{1+e}{1-e} \qquad \text{when} \qquad \varphi = \pm\frac{\pi}{2}$$

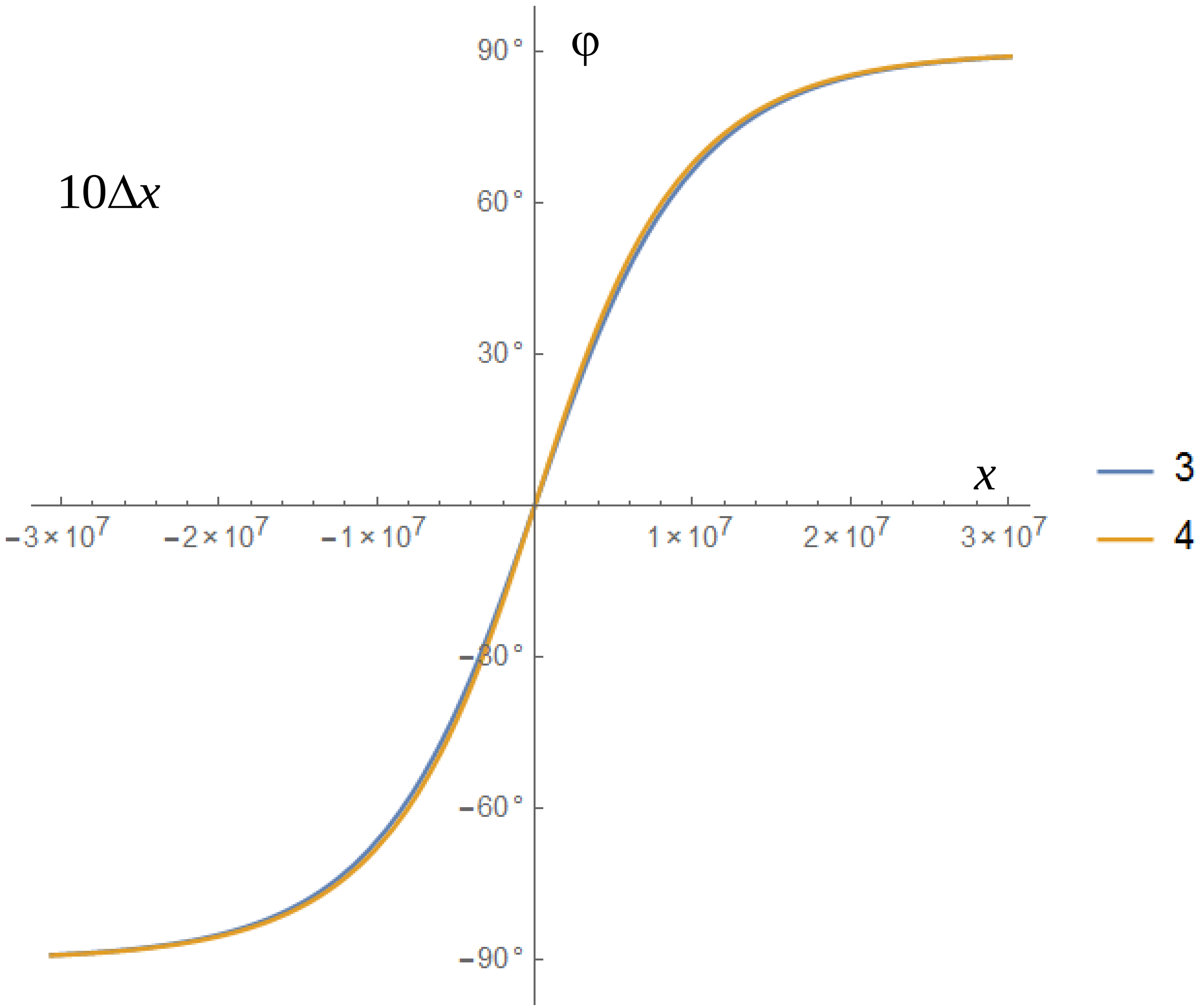
if $\alpha = 45°$ then $\max\left|\Delta y\right| = 42\,793\ \mathrm{m}$

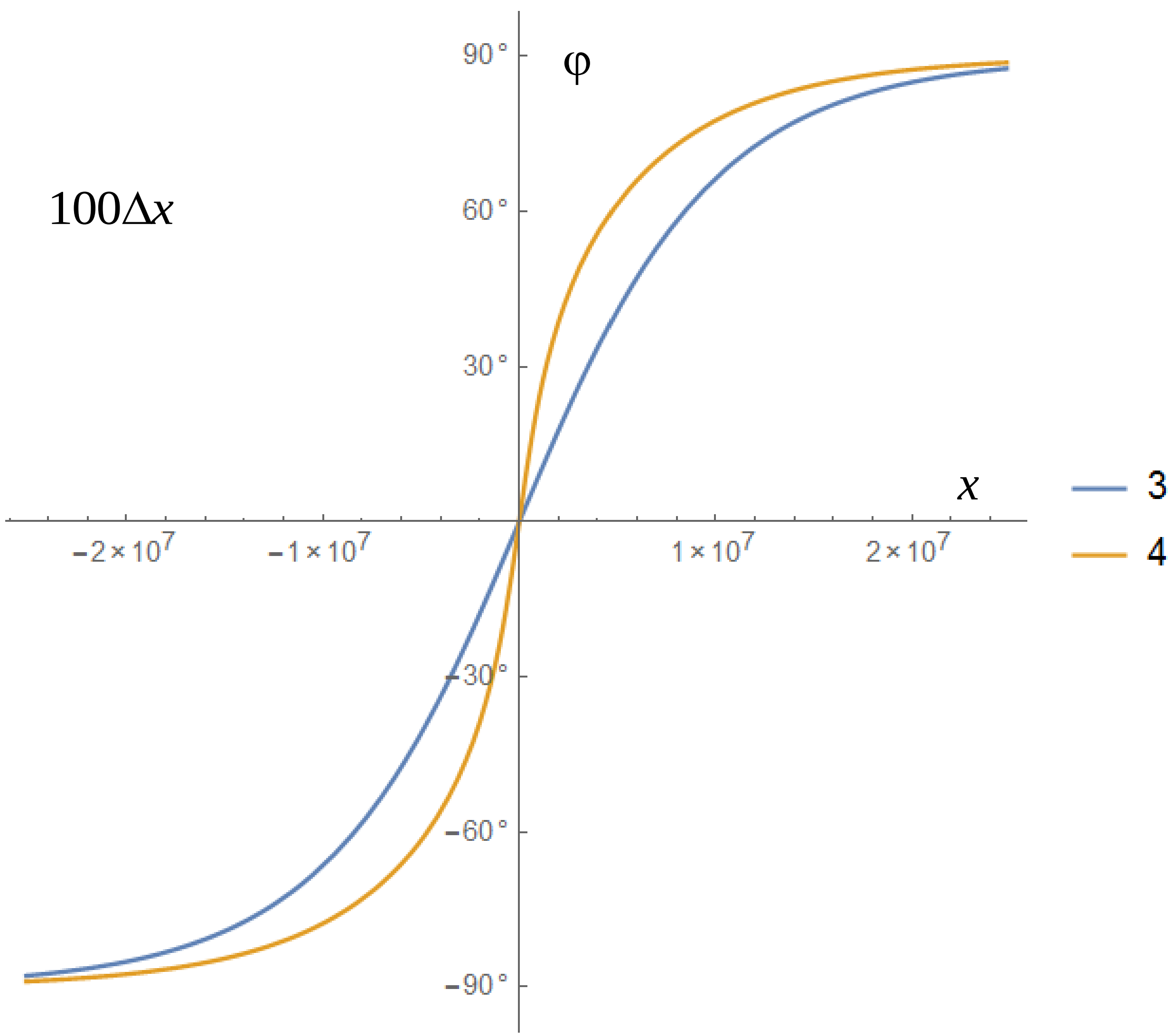
WGS84


y
1 × 10^7
10Δy
5 × 10^6
x
1
2
−1 × 10^7
−5 × 10^6
5 × 10^6
1 × 10^7
−5 × 10^6
−1 × 10^7



y
x
1×10^7
5×10^6
-5×10^6
-1×10^7
-1×10^7
-5×10^6
5×10^6
1×10^7
30 350 m
42 793 m
$100\Delta y$
1
2


10Δx
φ
90°
60°
30°
−30°
−60°
−90°
x
−3×10^7
−2×10^7
−1×10^7
1×10^7
2×10^7
3×10^7
3
4



100Δx
90°
60°
30°
−30°
−60°
−90°
φ
x
−2×10^7
−1×10^7
1×10^7
2×10^7
3
4

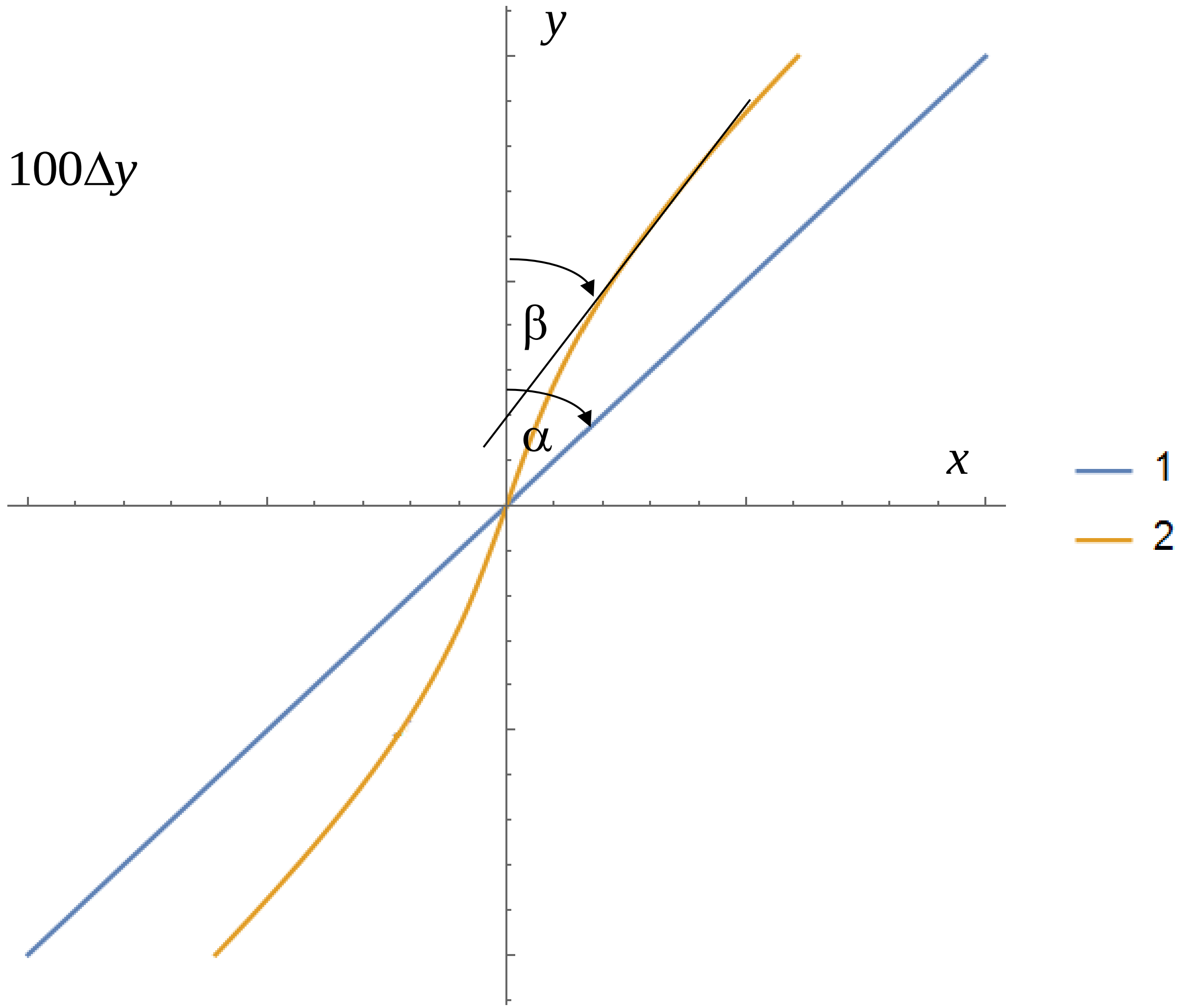
To compare the azimuths

$$x = \tan\alpha\ y \qquad dx = \tan\alpha\ dy$$

$$x = \tan\alpha\left[y - z(y)\right] \qquad dx = \tan\alpha\left(1 - \frac{dz}{dy}\right)dy = \tan\beta\, dy$$

$$\tan\beta = \tan\alpha\left(1 - \frac{dz}{dy}\right) = \tan\alpha\,\frac{1-e^2}{1-e^2\sin^2\varphi} < \tan\alpha$$

$$\beta - \alpha = \arctan\frac{-e^2\sin\alpha\cos\alpha\cos^2\varphi}{1-e^2(1-\sin^2\alpha\cos^2\varphi)}$$



100Δy
y
β
α
x
1
2

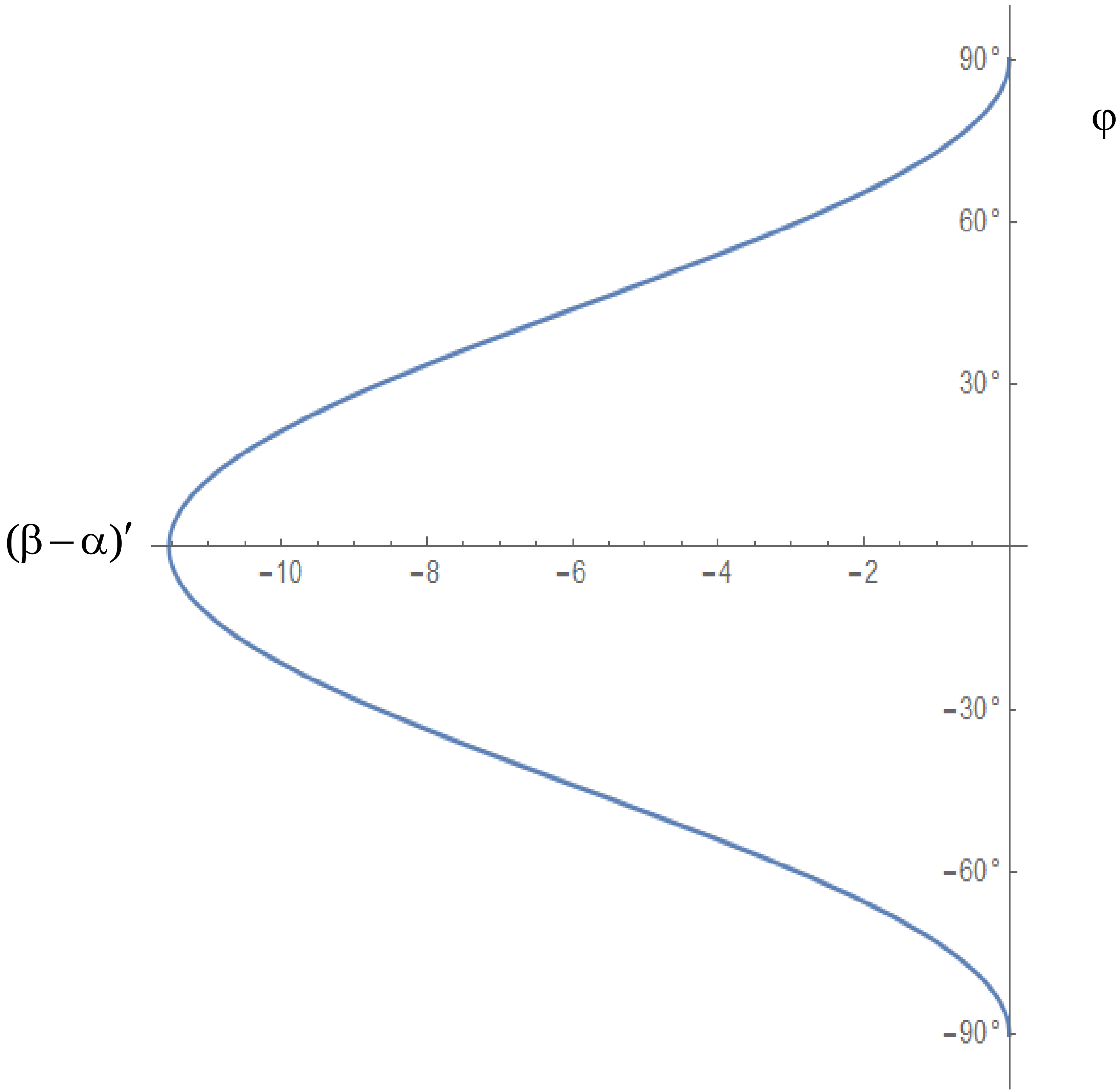
$$\beta-\alpha=\arctan\frac{-e^2\sin\alpha\cos\alpha\cos^2\varphi}{1-e^2(1-\sin^2\alpha\cos^2\varphi)}$$

$$\beta=\alpha \qquad \text{when} \qquad \varphi=\pm\frac{\pi}{2}$$

$$\max|\beta-\alpha|=\arctan\frac{e^2\sin\alpha\cos\alpha}{1-e^2\cos^2\alpha} \qquad \text{when} \qquad \varphi=0$$

$$\text{if} \quad \alpha=45° \quad \text{then} \quad \max|\beta-\alpha|=11'33''$$

WGS84


φ
90°
60°
30°
-30°
-60°
-90°
(β−α)′
-10
-8
-6
-4
-2

Thank you for you attention!